

AG-GROUPS AND OTHER CLASSES OF RIGHT BOL QUASIGROUPS

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ABSTRACT. By a result of Sharma, right Bol quasigroups are obtainable from right Bol loops via an involutive automorphism. We prove that the class of AG-groups, introduced by Kamran, is obtained via the same construction from abelian groups. We further introduce a new class of Bol* quasigroups, which turns out to correspond, as above, to the class of groups.

Sharma's correspondence allows an efficient implementation and we present some enumeration results for the above three classes.

1. INTRODUCTION

By definition, an AG-group (also called an LA-group) G is a set with a binary operation satisfying the left invertive law : $(xy)z = (zy)x$ for all $x, y, z \in G$ and also having left identity and left inverses. From these axioms it follows that the left inverse also is the right inverse and thus it becomes a two sided inverse. In particular, AG-groups belong to the class of quasigroups [10]. AG-groups were introduced in the PhD thesis of Kamran [2] and they first appeared in print in [3]. Some of the basic properties of AG-groups were discussed in [8]. For example, associativity, commutativity, and the existence of identity are equivalent properties for AG-groups. In particular, among the groups only the abelian groups are AG-groups. For a geometric interpretation of AG-groups see [12]. AG-groupoids (also called LA-semigroups), which generalize AG-groups, have applications in flock theory, see [6]. For additional sources on AG-groupoids, we suggest [4], and also [13]. It was noticed in [10] that AG-groups belong to the class of right Bol quasigroups. It is well known that right Bol quasigroups and right Bol loops have applications in differential geometry [7]. In [8] enumeration of AG-groups was proposed as an interesting problem. In [9] the enumeration was carried out computationally up to order 12. In this paper we completely classify AG-groups by showing that every AG-group arises from an abelian group via an involutive automorphism.

Theorem 1. *Suppose G is an abelian group and $\alpha \in \text{Aut}(G)$ satisfying $\alpha^2 = 1$. Define a new binary operation on G by $a \cdot b := \alpha(a) + b$. Then $G_\alpha = (G, \cdot)$ is an AG-group. Furthermore, every AG-group is obtainable in this way. Finally, the AG-groups G_α and H_β are isomorphic if and only if the abelian groups G and H are isomorphic and automorphisms α and β are conjugate.*

This description of the class of AG-groups allows us to classify various subclasses of them. For example, it easily follows from Theorem 1 that the AG-group G_α is a group if and only if α is the identity automorphism of the abelian group G . In

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the similar spirit let an AG-group be called *involutory* if its every element is an involution, that is, it satisfies $a^2 = e$, where e is the (left) identity element. The following is a corollary of Theorem 1.

Theorem 2. *An AG-group G_α is involutory if and only if α is the minus identity automorphism that is $\alpha(g) = -g$ for all $g \in G$. In particular, there is a natural bijection between abelian groups and involutory AG-groups.*

The groups of order one and two are the only cyclic groups for which the identity automorphism is the same as minus identity. In particular, for all orders $n > 2$ there exists a non-associative AG-group.

There have been a lot of publications (see for example, [1]) about the multiplication groups of loops and quasigroups. By definition, the multiplication group $M(Q)$ of a quasigroup Q is the subgroup of $\text{Sym}(Q)$ generated by all left and right translations. The multiplication group of an AG-group was studied in [11] where it was established that for a nonassociative AG-group of order n its multiplication group is nonabelian of order $2n$ and, correspondingly, the so called inner mapping group has order two. Based on Theorem 1, we can give a more precise description of the multiplication group.

Theorem 3. *Suppose G_α is a non-associative AG-group, that is, α is non-identity. Then $M(G_\alpha)$ is isomorphic to the semidirect product $G : \langle \alpha \rangle$. Note also that the order two group $\langle \alpha \rangle$ is the inner mapping group $I(G_\alpha)$, that is, the stabilizer in $M(G_\alpha)$ of the identity element.*

The construction of the AG-groups from the abelian groups, as described in Theorem 1, can easily be implemented in a computer algebra system. In fact, we implemented it in GAP [5] and were able to enumerate all AG-groups up to the order 2009. We stopped at the number because we used the small group library of GAP as our source of abelian groups. The method can easily be extended to much greater orders, as long as the abelian groups of that order are available. As a sample of the computation, we provide here (see Table 1) the information about the number of AG-groups up to order 20.

| Order | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-------|---|---|---|---|---|----|---|----|----|----|----|----|----|----|----|----|----|----|
| Group | 1 | 2 | 1 | 1 | 1 | 3 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 5 | 1 | 2 | 1 | 2 |
| Other | 1 | 2 | 1 | 1 | 1 | 7 | 3 | 1 | 1 | 6 | 1 | 1 | 3 | 24 | 1 | 3 | 1 | 6 |
| Total | 2 | 4 | 2 | 2 | 2 | 10 | 4 | 2 | 2 | 8 | 2 | 2 | 4 | 29 | 2 | 5 | 2 | 8 |

TABLE 1. Number of AG-groups of order n , $3 \leq n \leq 20$

The correspondence between the classes of abelian groups and AG-groups is very simple, so naturally, we were wondering whether a similar construction had been known. And indeed, we found a paper by Sharma [14] establishing a correspondence between the classes of left Bol loops and left Bol quasigroups. By duality, there is a similar correspondence between right Bol loops and right Bol quasigroups. This dual correspondence is essentially the same as our correspondence. Clearly, the class of abelian groups is a subclass of the class of right Bol loops. It is not so immediately clear, but still can be shown that the class of AG-groups is a subclass of the class of right Bol quasigroups. Hence our correspondence is simply a special

case of Sharma's correspondence adjusted for the case of right Bol loops. In this sense, our Theorem 1 shows that the class of AG-groups is the counterpart of the class of abelian groups under Sharma's correspondence. We consider it an interesting problem to determine which classes of quasigroups are the counterparts of other subclasses of right Bol loops, such as say, the class of groups or the class of Moufang loops. In this paper we give an answer to the first of these questions, namely, we provide the axioms for the class of quasigroups corresponding to the class of groups.

Definition 1. A right Bol* quasigroup is a quasigroup satisfying

$$a(bc \cdot d) = (ab \cdot c)d$$

for all elements a, b, c, d .

Note that the substitution $d = b$ turns the above equality into the right Bol law, which shows that the class of the right Bol* quasigroups is a subclass of right Bol quasigroups. In the future we will just speak of Bol quasigroups and Bol* quasigroups, skipping 'right'.

Theorem 4. Suppose G is a group and $\alpha \in \text{Aut}(G)$ satisfying $\alpha^2 = 1$. Define a new binary operation on G by $a * b := \alpha(a)b$. Then $G_\alpha = (G, *)$ is a Bol* quasigroup. Furthermore, every Bol* quasigroup is obtainable in this way. Finally, the Bol* quasigroups G_α and H_β are isomorphic if and only if the groups G and H are isomorphic and automorphisms α and β are conjugate.

2. PRELIMINARIES

The following property of AG-groups was established in [10].

Lemma 1. Every AG-group satisfies the identity $(ab \cdot c)d = a(bc \cdot d)$. In other words, every AG-group is a Bol* quasigroup.

We now embark on proving Theorem 1. We start with the first claim in that theorem.

Proposition 1. Let G be an abelian group under addition and let $\alpha \in \text{Aut}(G)$ be such that $\alpha^2 = 1$. Define $x \cdot y = \alpha(x) + y$ for all $x, y \in G$. Then $G_\alpha = (G, \cdot)$ is an AG-group with left identity $e = 0$.

Proof. We start by checking the left invertive law in G_α . Let $x, y, z \in G$. Then $xy \cdot z = \alpha(\alpha(x) + y) + z = \alpha^2(x) + \alpha(y) + z = x + \alpha(y) + z$, since $\alpha^2 = 1$. Similarly, $zy \cdot x = z + \alpha(y) + x$, and so $zy \cdot x = z + \alpha(y) + x = x + \alpha(y) + z = xy \cdot z$.

It is easy to see that 0 is the left identity in G_α . Indeed, $0x = \alpha(0) + x = 0 + x$, for all $x \in G$. Finally, we claim that $\alpha(-x)$ is the left inverse of x . Indeed, $\alpha(-x)x = \alpha(\alpha(-x)) + x = -x + x = 0$.

This shows that G_α is an AG-group. \square

We next need to show that every AG-group can be obtained as above. Let G be an AG-group with a left identity e . We first show how to build an abelian group from G .

Proposition 2. Consider the set G together with the new operation $+$ defined as follows:

$$x + y := xe \cdot y,$$

for all $x, y \in G$. Then $(G, +)$ is an abelian group. The zero element of this group is e and, for every $x \in G$, the inverse $-x$ is equal to $x^{-1}e$.

Proof. We start by checking associativity of addition. Let $x, y, z \in G$. Then $(x + y) + z = (xe \cdot y)e \cdot z$. Using Lemma 1 with $a = xe$, $b = y$, $c = e$, and $d = z$, we get that $(xe \cdot y)e \cdot z = xe \cdot (ye \cdot z) = x + (y + z)$. Therefore, $(x + y) + z = x + (y + z)$, proving associativity.

Commutativity of addition follows essentially by the definition. Indeed, $x + y = xe \cdot y = ye \cdot x = y + x$ by the left invertive law. Similarly, $e + x = ee \cdot x = ex = x$. Now by commutativity e is the zero element of $(G, +)$. Finally, $x^{-1}e + x = (x^{-1}e)e \cdot x = ((ee)x^{-1})x = ex^{-1} \cdot x = x^{-1}x = e$. Again, commutativity shows that $x^{-1}e$ is the inverse $-x$. \square

We remark that in place of the identity e we could use any fixed element $c \in G$. Namely, if we define addition via: $x \oplus y := xc \cdot y$ then we again get an abelian group, whose zero element is c and where the inverses are computed as follows: $\ominus x := x^{-1}c$. The proof is essentially the same. Furthermore, the groups obtained for different elements c are all isomorphic. Namely, the isomorphism between $(G, +)$ and (G, \oplus) is given by $x \mapsto x * c$.

Our next step is to prove that the mapping $\alpha : G \rightarrow G$ defined by $g \mapsto ge$ is an involutive automorphism of the abelian group $(G, +)$.

Proposition 3. *For all $x, y \in G$, we have $\alpha(x+y) = \alpha(x) + \alpha(y)$ and, furthermore, $\alpha^2 = 1$. Therefore, α is an involutive automorphism of $(G, +)$.*

Proof. We first note that by the left invertive law $\alpha^2(x) = xe \cdot e = ee \cdot x = ex = x$ for all $x \in G$. Therefore, $\alpha^2 = 1$, the identity mapping on G . By Lemma 1, $\alpha(x+y) = \alpha(xe \cdot y) = (xe \cdot y)e = x(ey \cdot e) = x(ye)$. On the other hand, $\alpha(x) + \alpha(y) = (xe)e \cdot ye$. We saw above that $xe \cdot e = x$, hence $\alpha(x) + \alpha(y) = x(ye)$, which we have shown to be equal to $\alpha(x+y)$. Therefore, α is an automorphism. \square

The last two results show that every AG-group G canonically defines an abelian group $(G, +)$ and its involutive automorphism α . It remains to see that the AG-group G can be recovered from $(G, +)$ and α as in Proposition 1.

Proposition 4. *Suppose G is an AG-group and let $(G, +)$ and α be the corresponding abelian group and its involutive automorphism. Then for all $x, y \in G$ we have $xy = \alpha(x) + y$. That is, $G = G_\alpha$.*

Proof. This is clear: indeed, $\alpha(x) + y = (xe \cdot e)y = xy$. We used the identity $xe \cdot e = x$, which we showed before. \square

We now turn to homomorphisms between AG-groups.

Proposition 5. *Suppose G and H are abelian groups and let $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$ and $\beta \in \text{Aut}(H)$ with $\beta^2 = 1$. Then the set of homomorphisms between AG-groups G_α and H_β coincides with the set of group homomorphisms $\pi : G \rightarrow H$ satisfying $\pi\alpha = \beta\pi$.*

Proof. Suppose $\pi : G_\alpha \rightarrow H_\beta$ is a homomorphism of AG-groups, that is, it is a mapping $G \rightarrow H$ such that $\pi(gh) = \pi(g)\pi(h)$. By cancellativity, π sends the left identity of G_α to the left identity of H_β . Therefore, for $x, y \in G$, we have $\pi(x+y) = \pi(xe \cdot y) = \pi(xe)\pi(y) = \pi(x)\pi(e) \cdot \pi(y) = \pi(x)e \cdot \pi(y) = \pi(x) + \pi(y)$. This shows that π is a homomorphism of abelian groups. Next, let $x \in G$. Then

$\pi\alpha(x) = \pi(xe) = \pi(x)e = \beta\pi(x)$. Since $x \in G$ is arbitrary, we conclude that $\pi\alpha = \beta\pi$.

For the converse, suppose that $\pi : G \rightarrow H$ is a homomorphism of abelian groups and that π satisfies $\pi\alpha = \beta\pi$. Then, for $x, y \in G$, we have $\pi(xy) = \pi(\alpha(x) + y) = \pi\alpha(x) + \pi(y) = \beta\pi(x) + \pi(y) = \pi(x)\pi(y)$. Hence π is a homomorphism of AG-groups. \square

This allows to complete the proof of Theorem 1.

Corollary 1. *Two AG-groups G_α and H_β are isomorphic if and only if there is an isomorphism π between G and H , satisfying $\pi\alpha\pi^{-1} = \beta$.*

Proof. Immediately follows from Proposition 5. Indeed, if π is bijective then the condition $\pi\alpha = \beta\pi$ is equivalent to $\pi\alpha\pi^{-1} = \beta$. \square

We record here a further corollary of Proposition 5, which describes the full automorphism group of the AG-group G_α .

Corollary 2. *The automorphism group of the AG-group G_α coincides with $C_{\text{Aut}(G)}(\alpha)$, the centralizer in $\text{Aut}(G)$ of the involution α .*

Proof. If $G_\alpha = H_\beta$ (and so $G = H$ and $\alpha = \beta$) then the condition $\pi\alpha = \beta\pi = \alpha\pi$ means simply that $\pi \in \text{Aut}(G)$ must commute with α . \square

It is interesting that the involutory twist construction can be used repeatedly.

Proposition 6. *Let (G, \circ) be an AG-group with a left identity e . Let $\alpha \in \text{Aut}(G)$ such that $\alpha^2 = 1$. Define $x \cdot y = \alpha(x) \circ y$ for all $x, y \in G$. Then (G, \cdot) is again an AG-group.*

Proof. Initially this had an independent proof. However, with all the theory that we have developed, this result follows easily. Indeed, by Theorem 1, the AG-group (G, \circ) must be equal to G_β , for an abelian group G and its involutory automorphism β .

Note that this means that $x \circ y = \beta(x) + y$, where, as usual, plus indicates addition in the abelian group G . According to Corollary 2, α is an automorphism of the group G commuting with β . In particular, $\gamma = \beta\alpha$ is again an involutory automorphism of G .

We now notice that $x \cdot y = \alpha(x) \circ y = \beta(\alpha(x)) + y = \gamma(x) + y$. This means that (G, \cdot) is simply the AG-group G_γ . \square

3. PARTICULAR CLASSES OF AG-GROUPS

It is natural to ask when the AG-group G_α is associative, that is, a group. It was shown in [8] that for AG-groups associativity is equivalent to commutativity and also to the property that the left identity e is a two-sided identity. We can show that, in fact, G_α is never a group, when $\alpha \neq 1$.

Proposition 7. *Suppose G is an abelian group and $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$. Then G_α is a group if and only if $\alpha = 1$.*

Proof. If $\alpha = 1$ then $\alpha(x) + y = x + y$, hence G_α is simply the group G . Conversely, assume that G_α is a group. Note that $e = 0$ is the left identity of G_α , since $0 \cdot x = \alpha(0) + x = 0 + x = x$. However, in a group the left identity is the same as the right identity. Therefore, for all $x \in G$, we must have $x \cdot 0 = x$. However, $x \cdot 0 = \alpha(x) + 0 = \alpha(x)$. Hence, $\alpha(x) = x$ for all $x \in G$, which means that $\alpha = 1$. \square

This proof already verifies that G_α is a group whenever it has a two-sided identity. Quite similarly, if G_α is commutative then for every $x \in G$ we have $x \cdot 0 = 0 \cdot x = x$. On the other hand, $x \cdot 0 = \alpha(x) + 0 = \alpha(x)$. Hence we must have that $\alpha(x) = x$ for all $x \in G$, and so $G_\alpha = G$ is a group. This shows that indeed commutativity is also equivalent to associativity.

The second interesting class of AG-groups is the class of involutory AG-groups. Recall from the introduction that an AG-group G is called involutory if its every nontrivial element is an involution, i.e., $x^2 = e$ for all $x \in G$ where e is the left identity of G .

Proposition 8. *Suppose G is an abelian group and $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$. Then G_α is an involutory if and only if $\alpha = -1$. (This means that $\alpha(x) = -x$ for all $x \in G$.)*

Proof. Recall that $x \cdot x = \alpha(x) + x$, so $x \cdot x = e = 0$ if and only if $\alpha(x) + x = 0$, that is, $\alpha(x) = -x$, so G_α is involutory if and only if $\alpha(x) = -x$ for all x . \square

As a consequence, we get the following.

Corollary 3. *For every order $n \geq 3$ there exists a non-associative AG-group of order n .*

Proof. Indeed, we can take $G = C_n$, the cyclic group of order n , and $\alpha = -1$. When $n \geq 3$, we have $\alpha \neq 1$, which means that G_α is non-associative by Proposition 7. \square

Since for every prime order $n = p > 2$ there exists exactly one abelian group, the cyclic group C_p , and since $\text{Aut}(C_p) \cong C_{p-1}$, which has a unique element of order two, we have the following result.

Corollary 4. *For every prime order $n = p \geq 3$, there is only one non-associative AG-group of order n .*

4. SOME EXAMPLES

For illustration we provide some examples. The case of the prime order has been dealt with in the preceding section.

Example 1. *For order 6, we have only one abelian group, namely C_6 . Since $\text{Aut}(C_6)$ has only one nontrivial involution, there are exactly two AG-groups of order 6, one associative, C_6 , and one non-associative, namely, $(C_6)_\alpha$, where $\alpha = -1$.*

The same is true for all orders $2p$, where p is an odd prime. So this case is similar to the case of the odd prime order.

Example 2. *For order 12, there are exactly two abelian groups, namely C_{12} and $C_6 \times C_2$. In the first case, $\text{Aut}(C_{12})$ is an elementary abelian group of order four. Hence its every element can be used to construct a new AG-group. This gives us four AG-groups (one associative, one non-associative involutory, and two further non-associative non-involutory). In the second case, $\text{Aut}(C_6 \times C_2)$ is isomorphic to $C_2 \times \text{Sym}(3)$, and so is nonabelian of order 12. In addition to the identity element, this group has three conjugacy classes of involutions. Hence, in this case, too, we get four different AG-groups.*

In total, we obtain eight AG-groups of order 12, out of which six are non-associative.

Example 3. Let us consider the order $2009 = 7^2 \cdot 41$. Again, there are two abelian groups of this order, C_{2009} and $C_{287} \times C_7$. In the first case the automorphism group is abelian, containing three involutions. Hence this group leads to four AG-groups. The automorphism group of $C_{287} \times C_7$ is isomorphic to $C_{40} \times GL(2, 7)$. This group has five conjugacy classes on involutions in addition to the identity element, hence in this case we obtain six different AG-groups.

In total, there are 10 different AG-groups of order 2009, out of which eight are non-associative.

5. A GAP PACKAGE FOR COMPUTING WITH AG-GROUPS

V. Sorge and the first author developed a GAP package AGGROUPOIDS which, in particular, contains functions dealing with AG-groups. They are based on the theory developed in this paper.

There are four main functions:

- The function `NrAllSmallNonassociativeAGGroups(n)` returns the total number of nonassociative AG-groups of order n provided that the `SmallGroups` library contains the list of groups of order n . This restriction will be lifted in the future, since all abelian groups of a given order are easy to construct.
- The function `AllSmallNonassociativeAGGroups(n)` returns the list of all non-associative AG-groups of the given order. Each AG-group is represented as a GAP quasigroup.
- The function `NrAllSmallNonassociativeAGGroupsFromAnAbelianGroup(G)` returns the total number of non-associative AG-groups that can be obtained from the abelian group G . This is equal to the number of conjugacy classes of involutions in $\text{Aut}(G)$.
- The function `AllSmallNonassociativeAGGroupsFromAnAbelianGroup(G)` returns the list of non-associative AG-groups obtainable from G , again as GAP quasigroups.

The entire package (not limited to these four functions) will shortly be available from the GAP repository.

6. MULTIPLICATION GROUP OF AN AG-GROUP

The concept of the multiplication group of a loop and, more generally, a quasigroup is well known. In a quasigroup Q , multiplication on the left (or right) by an element $x \in Q$ is a permutation L_x (respectively, R_x) of Q called the *left* (respectively, *right*) *translation* by x . The set of all left translations is called the *left section*, and similarly, the set of right translations is called the *right section* of Q . We will write `LSec` and `RSec` for the left and right sections, respectively. Therefore, $\text{LSec} = \{L_x \mid x \in Q\}$ and $\text{RSec} = \{R_x \mid x \in Q\}$.

The multiplication group $M(Q)$ is the subgroup of the symmetric group $\text{Sym}(Q)$ generated by $\text{LSec} \cup \text{RSec}$. If Q is a loop, the stabilizer in $M(Q)$ of the identity is called the *inner mapping group* and denoted $\text{Inn}(Q)$.

Since every AG-group G_α is a quasigroup we can consider its multiplication group $M(G_\alpha)$. Since G_α has a left identity 0, we can generalize the concept of the inner mapping group to the class of AG-groups by setting $\text{Inn}(G_\alpha)$ to be the stabilizer of 0 in $M(G_\alpha)$.

Proposition 9. *Let G be an abelian group and $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$. Then the following hold:*

- (1) $M(G_\alpha) = \text{LSec} \cup \text{RSec}$;
- (2) $\text{Inn}(G_\alpha) = \langle \alpha \rangle$;
- (3) LSec is a normal subgroup of $M(G_\alpha)$ and it is naturally isomorphic to G ;
- (4) $\text{RSec} = \alpha \text{LSec}$; and
- (5) $M(G_\alpha)$ is isomorphic to the semidirect product of G with the cyclic group $\langle \alpha \rangle$.

Proof. First note that the mapping $\psi : x \mapsto L_x$ is a homomorphism from G to $\text{Sym}(G)$. Indeed, $L_{x+y}(z) = (x+y) \cdot z = \alpha(x+y) + z = \alpha(x) + \alpha(y) + z = x \cdot (\alpha(y) + z) = x \cdot (y \cdot z) = L_x(L_y(z))$ for all $z \in G$. This means that L_{x+y} is indeed the product of L_x and L_y . Since ψ is a homomorphism, its image LSec is a subgroup of $\text{Sym}(G)$. Furthermore, if $L_x(z) = z$ for some $z \in G$ then $\alpha(x) + z = z$, which implies that $x = 0$. Therefore, ψ is injective and so it is an isomorphism from G onto LSec .

Next, note that $\alpha(z) = \alpha(z) + 0 = z \cdot 0 = R_0(z)$. This means that $\alpha = R_0$ is an element of RSec . Furthermore, $R_x(z) = \alpha(z) + x = \alpha(z) + \alpha^2(x) = \alpha(\alpha(x)) + \alpha(z) = \alpha(\alpha(x) + z) = (\alpha L_x)(z)$. This means that $R_x = \alpha L_x$ for all $x \in G$, that is, RSec is the coset of LSec containing α .

We now turn to part (1). We claim that α normalizes LSec . Indeed, $(\alpha L_x \alpha)(z) = \alpha L_x(\alpha(z)) = \alpha(\alpha(x) + \alpha(z)) = x + z = \alpha(\alpha(x)) + z = L_{\alpha(x)}(z)$. Thus, $\alpha L_x \alpha = L_{\alpha(x)}$, proving that α normalizes the subgroup LSec . Since $\text{RSec} = \alpha \text{LSec}$, we conclude that every element of RSec normalizes LSec , which means that LSec is normal in $M(G_\alpha)$. Also, it means that $M(G_\alpha) = \langle \text{LSec}, \alpha \rangle$, which implies that LSec has index at most two in $M(G_\alpha)$. (This proves (1).) To be more precise, the index is two if and only if $\alpha \notin \text{LSec}$. Clearly, α fixes 0 and, as we have already seen, the only element of LSec fixing 0 is L_0 , the identity element of LSec . Hence LSec has index two in $M(G_\alpha)$ if and only if $\alpha \neq 1$.

From the above, we also have that $|\text{Inn}(G_\alpha)| = |\alpha|$, since LSec is regular on G and so $|\text{Inn}(G_\alpha)|$ is equal to the index of LSec in $M(G_\alpha)$. Since α fixes 0, we have $\alpha \in \text{Inn}(G_\alpha)$, which implies (2). Parts (3) and (4) have already been proven. Finally, since $\alpha \notin \text{LSec}$ and $M(G_\alpha) = \langle \text{LSec}, \alpha \rangle$, (5) follows as well. \square

As an example of how the multiplication group can be used to identify the AG-group, we present the following result.

Theorem 5. *Suppose $M = M(G_\alpha)$ for a non-associative AG-group G_α and $M \cong D_{2n}$. Then either G is the Klein four-group (and so $n = 4$) or $G \cong C_n$ is cyclic. In the latter case $\alpha = -1$, and hence G_α is involutory.*

Proof. First of all, since G_α is non-associative, α is a nontrivial automorphism of G and so $n = |G| \geq 3$. By Theorem 9, the abelian group G is isomorphic to an index two subgroup of M . From this, it immediately follows that either $n = 4$ and G is the Klein four-group, or $n \geq 3$ is arbitrary and G is cyclic. Finally, in the cyclic case, since $M(G_\alpha)$ is isomorphic to the semidirect product of G and $\langle \alpha \rangle$, we conclude that α inverts every element of G and so $\alpha = -1$. \square

We also give a general characterization of all groups that arise as multiplication group of an AG-group.

Theorem 6. *A nonabelian group M is isomorphic to a multiplication group of some non-associative AG-group if and only if $M \cong T \rtimes R$ where T is abelian and $|R| = 2$.*

Proof. If $M = M(G_\alpha)$ then $M = G \rtimes \langle \alpha \rangle$ and so all the claimed properties hold. Conversely, suppose $M = T \rtimes R$ where T is abelian and $|R| = 2$. Let $\alpha \in \text{Aut}(T)$ be the automorphism induced by the generator of R on T . Then $M \cong M(T_\alpha)$ by Proposition 9 (5). \square

7. SHARMA'S CORRESPONDENCE

In his paper [8] from 1976 Sharma proved the following theorem. We recall that the identity $(ab \cdot c)b = a(bc \cdot b)$ is known as the right Bol identity. The loops (respectively, quasigroups) satisfying this identity are called the right Bol loops (respectively, right Bol quasigroups).

Theorem 7. *Suppose G is a right Bol loop and $\alpha \in \text{Aut}(G)$ satisfying $\alpha^2 = 1$. Define a new binary operation on G by $a * b := \alpha(a)b$. Then $G_\alpha = (G, *)$ is a right Bol quasigroup. Furthermore, every right Bol quasigroup is obtainable in this way. Finally, the right Bol quasigroups G_α and H_β are isomorphic if and only if the right Bol loops G and H are isomorphic and automorphisms α and β are conjugate.*

In reality Sharma proved the “left” version of this theorem, but we switched to the above, “right” version because it matches better our own results.

In particular, Sharma's theorem implies that every right Bol quasigroup automatically has a left identity element.

We note that Sharma's construction is essentially the same as ours, except it is done for a different, larger class of objects, the Bol loops instead of abelian groups. In other words, what we proved in Theorem 1 means simply that the class of AG-groups is the counterpart of the subclass of abelian groups under Sharma's correspondence. It would be interesting to ask what are the counterparts of other subclasses of Bol loops, such as, say, groups or Moufang loops. We leave the Moufang loops case as an open question, but we have an answer for the class of groups.

Recall from the introduction that by a *Bol** quasigroup we mean a quasigroup satisfying

$$a(bc \cdot d) = (ab \cdot c)d$$

for all a, b, c, d . Note that this is clearly a subclass of Bol quasigroups. In particular, every Bol* quasigroup automatically has a left identity element.

Theorem 8. *Suppose G is a group and $\alpha \in \text{Aut}(G)$ satisfying $\alpha^2 = 1$. Define a new binary operation on G by $a * b := \alpha(a)b$. Then $G_\alpha = (G, *)$ is a Bol* quasigroup. Furthermore, every Bol* quasigroup is obtainable in this way. Finally, the Bol* quasigroups G_α and H_β are isomorphic if and only if the groups G and H are isomorphic and automorphisms α and β are conjugate.*

Proof. Let us first see that G_α as above satisfies the identity

$$a * ((b * c) * d) = ((a * b) * c) * d.$$

Indeed, $a * ((b * c) * d) = \alpha(a)(\alpha(\alpha(b)c)d) = \alpha(a)\alpha^2(b)\alpha(c)d = \alpha(a)b\alpha(c)d$, since $\alpha^2 = 1$. Similarly, $((a * b) * c) * d = \alpha(\alpha(a)\alpha(b)c)d = \alpha^3(a)\alpha^2(b)\alpha(c)d = \alpha(a)b\alpha(c)d$. So the identity holds, proving that G_α is a Bol* quasigroup.

Conversely, assume that $(G, *)$ is a Bol^* quasigroup with left identity e . For $x \in G$, define $\alpha(x) = x * e$ and also, for $x, y \in G$, define $xy = \alpha(x)y$. We need to see that (1) G with this new product is a group; (2) α is an automorphism of this group of order two; and (3) $(G, *) = G_\alpha$.

First of all, for $x, y, z \in G$, $x(yz) = \alpha(x) * (\alpha(y) * z) = (x * e) * ((y * e) * z)$. By the identity, the latter is equal to $((x * e) * y) * e * z$ and this is equal to $(x * ((e * y) * e)) * z$. On the other hand, $(xy)z = \alpha(\alpha(x) * y) * z = (((x * e) * y) * e) * z$, so we have $x(yz) = (xy)z$ for all $x, y, z \in G$, proving that the new operation is associative. Cancelativity is clear, so we have an associative quasigroup, hence a group. Note that e is the identity element of the group, since $ex = \alpha(e) * x = (e * e) * x = e * x = x$.

For (2), we first need to show that α is a permutation of order two: $\alpha^2(x) * z = ((x * e) * e) * z = x * ((e * e) * z) = x * z$, and so by cancellativity, $\alpha^2(x) = x$. Thus, $\alpha^2 = 1$. To show that α is an automorphism, we compute: $\alpha(xy) = (xy) * e = (\alpha(x) * y) * e = ((x * e) * y) * e = x * ((e * y) * e) = x * (y * e)$ and $\alpha(x)\alpha(y) = \alpha(\alpha(x)) * \alpha(y) = x * (y * e)$. Thus, $\alpha(xy) = \alpha(x)\alpha(y)$. Finally, (3) is clear since $x * y = \alpha^2(x) * y = \alpha(x)y$. Hence $x * y = \alpha(x)y$, which means that $(G, *) = G_\alpha$.

For the final claim in the theorem, we note that the proofs of Proposition 5 and Corollary 2 depend neither on commutativity of the group operation, nor on the left invertive identity, so they fully apply in our present case. \square

The package AGGROUPOIDS mentioned above also contains functions enumerating Bol^* quasigroups and Bol quasigroups based on Theorem 8 and Sharma's Theorem 7. In Tables 2 and 3 we provide the counting for the Bol^* quasigroups and Bol quasigroups up to order 20 and 30, respectively.

| Order | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----------|---|---|---|---|---|----|---|----|----|----|----|----|----|-----|----|----|----|----|
| Group | 1 | 2 | 1 | 2 | 1 | 5 | 2 | 2 | 1 | 5 | 1 | 2 | 1 | 14 | 1 | 5 | 1 | 5 |
| Non-group | 1 | 2 | 1 | 2 | 1 | 12 | 3 | 2 | 1 | 14 | 1 | 2 | 3 | 88 | 1 | 9 | 1 | 13 |
| Total | 2 | 4 | 2 | 4 | 2 | 17 | 5 | 4 | 2 | 19 | 2 | 4 | 4 | 102 | 2 | 14 | 2 | 18 |

TABLE 2. Number of Bol^* quasigroups of order n , $3 \leq n \leq 20$

| Order | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|----------|---|---|---|---|---|----|---|----|----|----|----|----|----|----|
| Bol loop | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 5 |
| Other | 1 | 2 | 1 | 1 | 1 | 7 | 3 | 1 | 1 | 6 | 1 | 1 | 3 | 24 |
| Total | 2 | 4 | 2 | 2 | 2 | 10 | 4 | 2 | 2 | 8 | 2 | 2 | 4 | 29 |

| Order | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|----------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Bol loop | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 5 |
| Other | 1 | 2 | 1 | 1 | 1 | 7 | 3 | 1 | 1 | 6 | 1 | 1 | 3 | 24 |
| Total | 2 | 4 | 2 | 2 | 2 | 10 | 4 | 2 | 2 | 8 | 2 | 2 | 4 | 29 |

TABLE 3. Number of Bol quasigroups of order n , $3 \leq n \leq 30$

It might be worth mentioning that we can enumerate Bol^* quasigroups for much larger orders, as long as the list of groups of that order is available. For Bol quasigroups, we can only go up to the order 31, as the list of Bol loops of order 32 is an open problem.

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